Investigation of the Hecke Group G_5 and its Eisenstein Series

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Introduction

This is a report of my REU project at UCLA during the summer of 2007. I worked under John Leo, a graduate student of William Duke. My project was related to the research that John is doing for his thesis. In this report I will lay out an introduction to the problem I was assigned, and then describe my efforts to solve it.

My project was about the Fourier coefficients of the Eisenstein series for the group G_5. The group and the Eisenstein series are both generalizations of a simpler case. We begin with the modular group, a subset of the Möbius transformations on the extended complex plane. Möbius transformations are functions of the form $f(z) = \frac{az+b}{cz+d}$ that satisfy certain conditions, and the modular group is the subset with a, b, c, d integers. To this group is associated is the Eisenstein series. Its Fourier coefficients happen to have interesting number theoretic properties; for example, both the Riemann zeta function $\zeta(x)$ and the sum of divisors function $\sigma_k(n)$ appear in the coefficients.

We can generalize the modular group by modifying its generators slightly. The modular group is generated by the two transformations $S(\tau) = -1/\tau$ and $T(\tau) = \tau+1$. Then the Hecke group G_m is the group generated by $S(\tau) = -1/\tau$ and $T_m(\tau) = \tau + \lambda_m$, where $\lambda_m = 2\cos(\pi/m)$. Note that G_3 is just the modular group.

We can also generalize Eisenstein series by reinterpreting it. The normalized Eisenstein series of weight k for G_3 is

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{c,d\in\mathbb{Z}}{}' \frac{1}{(c\tau + d)^k}.$$

We reinterpret the $(c\tau + d)$ by considering it the denominator of elements of the modular group. Thus the normalized Eisenstein series of weight k for G_m is the same sum, made over cosets of G_m having the same denominator.

So we come to the problem of finding the Fourier coefficients of the Eisenstein series for G_5. There is a particular formula that expresses the Fourier coefficients in terms of a sum over double cosets of G_5. To find the coefficients, we need an explicit representation of the group that would enable us to perform the sum. This is a problem mathematicians have been working on since the 1950's. Most of my project dealt with trying to find such a representation of G_5.

Möbius Transformations

Definition 1. A Möbius tranformation is a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$f(z) = \frac{az + b}{cz + d}$$

where a, b, c, and d are complex numbers satisfying $ad - bc \neq 0$. ([Apo90], p. 26)
Since f is a function on the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, the definition of f must be extended by

$$f\left(\frac{-d}{c}\right) = \infty \quad \text{and} \quad f(\infty) = \frac{a}{c} \quad \text{when } c \neq 0; \text{ when } c = 0, \quad f(\infty) = \infty.$$

Since multiplying a, b, c, and d all by the same constant does not affect anything, we can assume without loss of generality that $ad - bc = 1$.

Möbius transformations are automorphisms of $\hat{\mathbb{C}}$: bijective and holomorphic, with holomorphic inverse.

Möbius transformations have the following properties: if $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{Az+B}{Cz+D}$, then

$$f^{-1}(z) = \frac{dz - b}{-cz + a}$$

and

$$f(g(z)) = \frac{(aA + bC)z + (aB + bD)}{(cA + dC)z + (cB + dD)}.$$

Hence it is natural to associate to the function $f(z) = \frac{az+b}{cz+d}$ the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then if f is associated to the matrix A and g is associated to the matrix B, the composition $f \circ g$ is associated to the matrix product AB and f^{-1} is associated to the matrix inverse A^{-1}. The identity transformation $\frac{1z+0}{0z+1}$ is associated to the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Modular Group

We are interested in a specific subset of the Möbius transformations.

Definition 2. The modular group, denoted by Γ, is the set of Möbius transformations of the form

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

where a, b, c, d are integers such that $ad - bc = 1$. It can be represented by $SL_2(\mathbb{Z})$, the group of 2×2 integer matrices with determinant 1 modulo the relation $A \sim -A$. ([Apo90], p. 28)

Let $\tau' = \frac{a\tau+b}{c\tau+d}$. Then

$$\text{Im }(\tau') = \text{Im }\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{ad - bc}{|c\tau + d|^2}\text{Im }(\tau) = \frac{\text{Im }(\tau)}{|c\tau + d|^2}.$$

Hence τ' and τ have the same imaginary part, and $\tau' \in$ H, the upper half plane, if and only if $\tau \in$ H. Therefore we can consider Γ as acting on the upper half plane.

Theorem 3. *The modular group is generated by two matrices*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, an arbitrary element of Γ *is of the form*

$$T^{a_1} S T^{a_2} S \cdots S T^{a_k}$$

where the a_i *are integers. This representation is not unique.*

For a proof of the theorem, see [Apo90], p. 28. The generators S and T satisfy the relations $S^2 = I$ and $(ST)^3 = I$. These are the only relations on the generators. ([Ros54])

Eisenstein Series

Definition 4. Given an integer k, a meromorphic function $f : H \to \mathbb{C}$ is weakly modular of weight k if

$$f(A(\tau)) = (c\tau + d)^k f(\tau) \ \text{ for } \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ \text{ and } \ \tau \in H.$$

Definition 5. Given an integer k, a function $f : H \to \mathbb{C}$ is a modular form of weight k if

1. f is holmorphic on H,
2. f is weakly modular of weight k,
3. f is holomorphic at ∞.

For an explanation of what it means to be holomorphic at ∞, see [DS05], p. 3.

The zero function is a modular form of weight k for all k, and any constant function is a modular form of weight 0. The following is a nontrivial example of a modular form.

Definition 6. Let $k > 2$ be an even integer. Then the Eisenstein series of weight k for $\Gamma = SL_2(\mathbb{Z})$ is

$$G_k(\tau) = \sum_{c,d \in \mathbb{Z}}{}' \frac{1}{(c\tau + d)^k}, \qquad \tau \in H,$$

where the primed summation sign indicates that the sum is over pairs $(c, d) \in \mathbb{Z}^2 - (0, 0)$.

We are interested in obtaining the coefficients of the Fourier series for $G_k(\tau)$.
We begin by considering the identities

$$\frac{1}{\tau}+\sum_{d=1}^{\infty}\left(\frac{1}{\tau-d}+\frac{1}{\tau+d}\right)=\pi\cot\pi\tau=\pi i-2\pi i\sum_{m=0}^{\infty}q^m, q=e^{2\pi i\tau} \tag{1}$$

Taking $k-1$ derivatives of (1) with respect to τ gives

$$\sum_{d\in\mathbb{Z}}\frac{1}{(\tau+d)^k}=\frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}m^{k-1}q^m, \quad k\geq 2. \tag{2}$$

For $k>2$ even,

$$\sum_{c,d\in\mathbb{Z}}{}^{\prime}\frac{1}{(c\tau+d)^k}=\sum_{d\neq 0}\frac{1}{d^k}+2\sum_{c=1}^{\infty}\left(\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)^k}\right). \tag{3}$$

Let ζ denote the Riemann zeta function and use (2) to get

$$\sum_{c,d\in\mathbb{Z}}{}^{\prime}\frac{1}{(c\tau+d)^k}=2\zeta(k)+2\frac{(2\pi i)^k}{(k-1)!}\sum_{c=1}^{\infty}\sum_{m=1}^{\infty}m^{k-1}q^{cm}. \tag{4}$$

This can be rearranged to give the Fourier expansion

$$G_k(\tau)=2\zeta(k)+2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \quad k>2, k\text{even} \tag{5}$$

where $\sigma_k(n)$ is the sum of divisors function

$$\sigma_k(n)=\sum_{\substack{d|n\\ d>0}}d^k.$$

We can divide out the $2\zeta(k)$ to get a series having rational coefficients with a common denominator. ([DS05], p. 5,6)

Definition 7. The normalized Eisenstein series of weight k for $\Gamma=SL_2(\mathbb{Z})$ is

$$E_k(\tau)=G_k(\tau)/2\zeta(k)=\frac{1}{2\zeta(k)}\sum_{c,d\in\mathbb{Z}}{}^{\prime}\frac{1}{(c\tau+d)^k}=\frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)=1}}\frac{1}{(c\tau+d)^k}=1+\frac{(2\pi i)^k}{\zeta(k)(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

Hecke Groups

We can generalize the modular group, which was generated by the Möbius transformations

$$S(z)=\frac{-1}{z}, T(z)=z+1.$$

These new groups have two generators. One is $S(z)$. However, $T(z)$ is replaced by $T_m(z)=z+\lambda_m$.

Definition 8. The Hecke group G_m is the group of Möbius transformations acting on H generated by the two transformations

$$S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\quad\text{and}\quad T=\begin{pmatrix}1&\lambda_m\\0&1\end{pmatrix},\quad\text{where }\lambda_m=2\cos\left(\frac{\pi}{m}\right).$$

([Leo08])

Note: Henceforth, G_m shall refer to the Hecke group; Eisenstein series shall be denoted by $E_k(\tau)$. When it is clear which group we are working with, we will write T instead of T_m.

Since $\lambda_3=2\cos\left(\frac{\pi}{3}\right)=1, G_3$ is merely the modular group $SL_2(\mathbb{Z})$.

The generators S and T of G_m satisfy the relations $S^2=I$ and $(ST)^m=I$. These are the only relations on the generators.

Eisenstein Series for Hecke Groups

The Eisenstein series $E_k(\tau)$ for G_3 was a modular form of weight k, and hence satisfied weak modularity of weight k:

$$E_k(A(\tau)) = (c\tau+d)^k E_k(\tau) \;\; for \;\; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\; and \;\; \tau \in H.$$

In order to generalize the Eisenstein series for other Hecke groups, we must interpret it in another way. By the definition of Möbius transformations on $\hat{\mathbb{C}}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ sends ∞ to ∞ when $c = 0$. Hence we let $\Gamma_\infty = \langle T_3 \rangle$ be the subgroup of G_3 that stabilizes ∞. If $A \in G_3$, then the right coset $\Gamma_\infty A$ will contain matrices that all have the same bottom row. If we let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} x & y \\ c & d \end{smallmatrix}\right)$, $BA^{-1} = \left(\begin{smallmatrix} xd-yc & ay-bx \\ 0 & ad-bc \end{smallmatrix}\right)$ is upper triangular, meaning that $BA^{-1} \in \Gamma_\infty$, and hence implying that A and B are in the same coset. Also, if $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} x & y \\ -c & -d \end{smallmatrix}\right)$, A and B will be in the same coset since $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} -a & -b \\ -c & -d \end{smallmatrix}\right)$, which has the same bottom row as B.

Let $j_A(\tau) = c\tau + d$ for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ (the function satisfies $j_{AB}(\tau) = j_A(B(\tau))j_B(\tau)$.) Then we will have

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(c\tau+d)^k} = \sum_{A\in\Gamma_\infty\backslash G_3} \frac{1}{j_A(\tau)^k}.$$

We shall use the latter expression to define our generalization of the Eisenstein series Hecke groups.

Definition 9. The normalized Eisenstein series of weight k for the Hecke group G_m is

$$E_k(\tau) = \sum_{A\in\Gamma_\infty\backslash G_m} \frac{1}{j_A(\tau)^k}.$$

([Leo08])

For $B \in G_m$,

$$\begin{aligned} E_k(B\tau) &= \sum_{A\in\Gamma_\infty\backslash G_3} \frac{1}{j_A(B\tau)^k} && (6) \\ &= j_B(\tau)^k \sum_{A\in\Gamma_\infty\backslash G_3} \frac{1}{j_{AB}(\tau)^k} && (7) \\ &= j_B(\tau)^k E_k(\tau). && (8) \end{aligned}$$

which shows that E_k is still a modular form of weight k.

As before, we are interested in finding the coefficients of the Fourier series for $E_k(\tau)$. It is shown in [Iwa97] that

$$E_k(\tau) = 1 + \sum_{n=1}^{\infty} a_n q^n$$

where the coefficients are

$$a_n = \left(\frac{2\pi}{\lambda i}\right)^k \frac{n^{k-1}}{(k-1)!} \sum_{c>0} c^{-k} \sum_{A=\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash G_m/\Gamma_\infty} e\left(\frac{nd}{c\lambda}\right).$$

We shall use this formula to find the Fourier coeficients of the cases $m = 3, 4, 6$.

The Case $m = 3$

To find the coefficients a_n, we have to sum over double cosets. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We want $\Gamma_\infty \setminus A/\Gamma_\infty$. It would consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+xc & y(a+xc)+b+xd \\ c & d+cy \end{pmatrix},$$

where x and y vary over $\mathbb{Z}$.

Given c, we want d's that will result in each being a distinct double coset. Since the matrices have determinant 1, we know that $ad - bc = 1$. Hence, by Bezout's Lemma, c and d must be relatively prime. Also, because the double coset consisted of matrices with bottom row $\begin{pmatrix} c & d + cy \end{pmatrix}$, we know the d's must be distinct mod c. Hence we can choose all the $d \in (\mathbb{Z}/c\mathbb{Z})^{\times}$.

Ignoring the factors in front of the sum, we are looking for

$$S = \sum_{c>0} c^{-k} \sum_{A=\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash G_m / \Gamma_\infty} e\left(\frac{nd}{c}\right)$$

$$S = \sum_{c>0} c^{-k} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^{\times}} e\left(\frac{nd}{c}\right).$$

The expression

$$\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^{\times}} e\left(\frac{nd}{c}\right)$$

is called a Ramanujan sum and is known ([Wik07]) to be equal to

$$\sum_{\ell | (c,n)} \mu\left(\frac{c}{\ell}\right) \ell,$$

where $\mu(x)$ is the Möbius function. We relabel $d = \ell$ and continue with the original double sum.

$$S = \sum_{c>0} c^{-k} \sum_{d|(c,n)} \mu\left(\frac{c}{d}\right) d$$

$$S = \sum_{c>0} c^{-k} \sum_{d|n)} f(c,d),$$

where $f(c,d) = \begin{cases} \mu\left(\frac{c}{d}\right) d & \text{if } d|c \\ 0 & \text{otherwise.} \end{cases}$

Switching the order of the sums and moving the $\mu\left(\frac{c}{d}\right)$ over, we get

$$S = \sum_{d|n} d \sum_{\substack{c>0 \\ d|c}} c^{-k} \mu\left(\frac{c}{d}\right).$$

Now we substitute $c = dC$, and continue to get

$$S = \sum_{d|n} d \sum_{C>0} (dC)^{-k} \mu(C)$$

$$S = \sum_{d|n} d^{1-k} \sum_{C>0} C^{-k} \mu(C).$$

Now we show that

$$\sum_{C>0} C^{-k} \mu(C) = \frac{1}{\zeta(k)}.$$

The zeta function is defined by

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k} = \prod_p \frac{1}{1 - \frac{1}{p^k}}.$$

Hence we have

$$\begin{aligned}\frac{1}{\zeta(k)} &= \prod_p \left(1-\frac{1}{p^k}\right)\\ &= \left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{3^k}\right)\left(1-\frac{1}{5^k}\right)\cdots\\ &= 1-\sum_p \frac{1}{p^k}+\sum_{i<j}\frac{1}{{p_i}^k{p_j}^k}-\sum_{i<j<l}\frac{1}{{p_i}^k{p_j}^k{p_l}^k}\cdots\\ &= \sum_{C>0} C^{-k}\mu(C),\end{aligned}$$

as desired.

Bringing back the coefficients from our original sum, we have

$$S' = \left(\frac{2\pi}{i}\right)^k \frac{n^{k-1}}{(k-1)!}\sum_{d|n}\frac{d^{1-k}}{\zeta(k)}$$

$$S' = \frac{(2\pi i)^k}{\zeta(k)(k-1)!}\sum_{d|n}\left(\frac{n}{d}\right)^{k-1}$$

$$S' = \frac{(2\pi i)^k}{\zeta(k)(k-1)!}\sigma_{k-1}(n)$$

since as d ranges over the divisors of n, so does $\frac{n}{d}$. This is precisely the result we got earlier using the cotangent identities.

The Cases $m = 4, 6$

G_m was the group of Möbius transformations generated by $S(\tau) = \frac{-1}{\tau}$ and $T_m(\tau) = \tau + \lambda_m$, where $\lambda_m = 2\cos\left(\frac{\pi}{m}\right)$. $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$. Elements of both G_4 and G_6 are known ([Hut02],[You04]) to be those matrices of $SL_2(\mathbb{R})$ of the form

$$\begin{pmatrix} a\lambda & b \\ c & d\lambda \end{pmatrix} \text{ and } \begin{pmatrix} a & b\lambda \\ c\lambda & d \end{pmatrix},$$

where a, b, c, d are all integers. The determinants of the two types of matrices are $\lambda^2 ad - bc$ and $ad - \lambda^2 bc$, respectively. Hence, given relatively prime integers c and d, we have a matrix of the first type iff $\lambda^2 \nmid c$, and we have a matrix of the second type iff $\lambda^2 \nmid d$.

Let $M = \lambda^2$. Then the Fourier coefficients of the Eisenstein series of weight k for G_4 and G_6 are

$$a_n = \left(\frac{2\pi}{\lambda i}\right)^k \frac{n^{k-1}}{(k-1)!}\left(\sum_{\substack{c\in\mathbb{Z}^+\\(c,M)=1}} c^{-k}\sum_{\substack{0\le d<c\\(c,d)=1}} e\left(\frac{nd}{c}\right)+\sum_{c\in\mathbb{Z}^+}(c\lambda)^{-k}\sum_{\substack{0\le d<Mc\\(Mc,d)=1}} e\left(\frac{nd}{Mc}\right)\right).$$

Given positive integers c, n, M, we again have the Ramanujan sum

$$\sum_{\substack{0\le d<Mc\\(Mc,d)=1}} e\left(\frac{nd}{Mc}\right) = \sum_{d|(Mc,n)} \mu(Mc/d)d.$$

Then for M prime we have

$$\begin{aligned}
&\sum_{\substack{c\in\mathbb{Z}^+\\(c,M)=1}} c^{-k} \sum_{\substack{0\le d<c\\(c,d)=1}} e\left(\frac{nd}{c}\right) + \sum_{c\in\mathbb{Z}^+} (c\lambda)^{-k} \sum_{\substack{0\le d<Mc\\(Mc,d)=1}} e\left(\frac{nd}{Mc}\right)\\
&\quad= \sum_{\substack{c\in\mathbb{Z}^+\\(c,M)=1}} c^{-k} \sum_{d|(c,n)} \mu(c/d)d + \lambda^{-k} \sum_{c\in\mathbb{Z}^+} c^{-k} \sum_{d|(Mc,n)} \mu(Mc/d)d\\
&\quad= \sum_{d|n} d \sum_{\substack{c>0,(c,M)=1\\d|c}} c^{-k}\mu(c/d) + \lambda^{-k} \sum_{d|n} d \sum_{\substack{c>0\\d|Mc}} c^{-k}\mu(Mc/d)\\
&\quad= \sum_{d|n} d^{1-k} \sum_{\substack{C>0\\(dC,M)=1}} C^{-k}\mu(C) + \lambda^{-k} \sum_{d|n} d^{1-k} M^k \sum_{\substack{C>0\\M|dC}} C^{-k}\mu(C)\\
&\quad= \sum_{d|n} d^{1-k} \sum_{C>0} C^{-k}\mu(C) + (\lambda^k-1) \sum_{d|n} d^{1-k} \sum_{\substack{C>0\\M|dC}} C^{-k}\mu(C)\\
&\quad= \frac{\sigma_{k-1}(n)}{n^{k-1}\zeta(k)} + (\lambda^k-1) \sum_{d|n} d^{1-k} \sum_{\substack{C>0\\M|dC}} C^{-k}\mu(C)\\
&\quad= \frac{\sigma_{k-1}(n)}{n^{k-1}\zeta(k)} + (\lambda^k-1) \left(\sum_{\substack{d|n\\M|d}} d^{1-k} \frac{1}{\zeta(k)} + \sum_{\substack{d|n\\M\nmid d}} d^{1-k} \sum_{\substack{C>0\\M|C}} C^{-k}\mu(C) \right)\\
&\quad= \frac{\sigma_{k-1}(n)}{n^{k-1}\zeta(k)} + b_{n,M}(\lambda^k-1)\frac{\sigma_{k-1}(n/M)}{n^{k-1}\zeta(k)} + (\lambda^k-1)\frac{\sigma_{k-1}(N)}{N^{k-1}(1-M^k)\zeta(k)}\\
&\quad= \frac{\sigma_{k-1}(n)}{n^{k-1}\zeta(k)} + b_{n,M}(\lambda^k-1)\frac{\sigma_{k-1}(n/M)}{n^{k-1}\zeta(k)} - \frac{\sigma_{k-1}(N)}{(\lambda^k+1)N^{k-1}\zeta(k)}
\end{aligned}$$

where $N = n/M^{v_M(N)}$, $b_{n,M} = 1$ if $M|n$, 0 otherwise, and we substituted $C = c/d$ and $C = Mc/d$ in the fourth line. To determine $X = \sum_{\substack{C>0\\M|C}} C^{-k}\mu(C)$, let $Y = \sum_{\substack{C>0\\M\nmid C}} C^{-k}\mu(C)$. Then we have $\zeta(k)^{-1} = X + Y$, and $X = -M^{-k}Y$ for M prime. Solving for X gives us $X = ((1-M^k)\zeta(k))^{-1}$. ([Leo08])

The Case $m = 5$

When $m = 5$, $\lambda_m = \lambda_5 = \lambda = 2\cos\left(\frac{\pi}{m}\right) = \frac{1+\sqrt{5}}{2}$. λ is also called the golden ratio, and is a root of the quadratic polynomial $t^2 - t - 1$. Hence it satisfies $\lambda^2 = \lambda + 1$

My project was specifically on case $m = 5$. Like before, I was to find a way to determine the Fourier coefficients of the Eisenstein series for the group G_5. Because of the double coset formula for the coefficients,

$$a_n = \left(\frac{2\pi}{\lambda i}\right)^k \frac{n^{k-1}}{(k-1)!} \sum_{c>0} c^{-k} \sum_{A=\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash G_m/\Gamma_\infty} e\left(\frac{nd}{c\lambda}\right),$$

the task could be done if there were a way to perform the sum over the double cosets. In order to do this, however, I would have to find a systematic way of representing the elements of the group G_5. I attempted to do this in two ways: first I tried the obvious way, which was to look for a pattern in the matrices. Secondly, I attempted some form of word reduction. When neither of these attempts proved successful, I then tried two new approaches. To begin, I looked at the work of David Rosen, hoping some of his work on the Hecke groups G_m would be useful, which it ultimately was not. In my final attempt at this problem, I tried a method that was derived from the work of Armin Leutbecher. This too was unsuccessful.

The Obvious Approach

Recall that an arbitrary element of G_5 is of the form

$$T^{a_1}ST^{a_2}S\cdots ST^{a_k}.$$

With this in mind, I began my search for a way of indexing G_5. The obvious approach seemed to be to write out arbitrary words of increasing length and hope to find a pattern. I multiplied out the matrices, then used the relation $\lambda^2 = \lambda + 1$ to rewrite the entries. The following are the first four arbitrary words in S and T.

$$\begin{aligned}
T^a &= \begin{pmatrix} 1 & a\lambda \\ 0 & 1 \end{pmatrix} \\
T^aST^b &= \begin{pmatrix} a\lambda & ab\lambda + ab - 1 \\ 1 & b\lambda \end{pmatrix} \\
T^aST^bST^c &= \begin{pmatrix} ab\lambda + ab - 1 & (2abc - a - c)\lambda + abc \\ b\lambda & bc\lambda + bc - 1 \end{pmatrix} \\
T^aST^bST^cST^d &= \begin{pmatrix} (2abc - c - a)\lambda + abc & (3abcd - cd - ad - ab)\lambda + 2abcd - cd - ad - ab + 1 \\ bc\lambda + bc - 1 & (2bcd - d - b)\lambda + bcd \end{pmatrix}
\end{aligned}$$

Clearly, the matrix entries grow rapidly in both length and complexity. I was not able to find any pattern in the entries, and gave up on this approach.

Word Reduction

In my next approach, I looked for some method of rewriting an arbitrary element with the hope that I might somehow simplify that element. At the time, I was not aware that there were only two relations on the generators of G_5. Because this is the case, my attempts were utterly futile. The following examples are provided in order to give an idea of what my attempt was aiming for.

One hope I had was to somehow separate the T's and S's by moving them to opposite ends of the word. This idea was inspired by the dihedral group, D_n, the symmetries of the regular n-gon in the plane. D_n is generated by two elements: r, a rotation, and f, a flip. They satisfy the relations $f^2 = e$ and $frf = r^{n-1} = r^{-1}$. The latter can be written as $fr = r^{-1}f$. Hence, we have that $fr^k = r^{-k}f$. An element of D_n is just a word in r and f, but using the two relations, any element can be written as either r^k or r^kf, where $k \in \{1, 2, \dots n\}$. One merely moves the f's all the way to the right of the word. When an f meets another f, they cancel; when an f meets a power of r, we just flip them and negate the power.

I thought perhaps a similar approach might work with G_5. And so to move an S or a T, I let X and Y be the matrices satisfying $TS = STX$ and $ST = TSY$. Then we have

$$\begin{aligned}
X &= T^{-1}S^{-1}TS & \text{and} \qquad Y &= S^{-1}T^{-1}ST \\
&= T^{-1}STS & &= ST^{-1}ST,
\end{aligned}$$

Hence $X = [T, S]$ and $Y = [S, T]$ are commutators. We also have that $SX = YS$, and hence $SXS = Y$ and $SYS = X$. Clearly, $X^{-1} = Y$.

I proceeded to look for some relations on X and Y, fooling around with some calculations. For example, I wrote out $X^2 = T^{-1}STS^{-1}STS$. I then flipped an S and a T twice, resulting in $X^2 = T^{-1}SSTXT^{-1}SSTX = T^{-1}TXT^{-1}TX = X^2$. All calculations of this sort either reduce to something trivial, like this example did, or just end up doing nothing beyond writing out expressions for an object in T and S that are precisely what one could have gotten by messing with the S's and T's.

Since an arbitrary word has powers of T, I checked to see how I might flip the expression T^kS. Well, $T^kS = T^{k-1}TS = T^{k-1}STX = T^{k-2}TSTX = T^{k-2}STXTX = T^{k-2}S(TX)^2 = \cdots = T^0S(TX)^k = S(TX)^k$. It would seem that I just did something significant, but, as before, nothing has changed at all. For, if we ask what $(TX)^k$ is, we see that $(TX)^k = (TT^{-1}STS)^k = (STS)^k = ST^kS$. So all we have actually done is write $T^kS = S(TX)^k = SST^kS = T^kS$.

Without any additional relations, this approach was useless. While G_5 and D_n superficially look somewhat alike- both have two generators and two relations, one of which says the square of one of the generators is the identity- there exists a key difference between the two: D_n has a relation that involves flipping the generators. G_5's second relation does not help in that; it merely allows one to reduce powers of (ST) mod 5. Hence, this method's only accomplishments were to insert pairs like TT^{-1} or SS into the word in several places, and then use associativity to group things differently. These groupings perhaps made a word appear shorter, but they were not doing anything relevant or helpful. Unfortunately, I did not find out about the relations on the generators of G_5 until I began my next approach, and hence continued with these unsuccessful calculations for quite a while.

Rosen and Continued Fractions

David Rosen has worked on the Hecke groups G_m, considering their connection with continued fractions. The connection arises naturally, since the element $T^{a_1}ST^{a_2}S\cdots ST^{a_k}$ sends z to

$$= a_1\lambda - \cfrac{1}{a_2\lambda - \cfrac{1}{a_3\lambda - \cfrac{\cdots}{a_k\lambda + z}}}$$

We will denote the continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}}$$

as $(b_0, a_1/b_1, a_2/b_2, ...)$

If we cut off the continued fraction after $n+1$ terms, we get a finite continued fraction that can be written as a quotient of two polynomials in a_i and b_i, and denoted $(b_0, a_1/b_1, a_2/b_2, ..., a_n/b_n) = P_n/Q_n$.

Rosen proved theorems that determine when a Möbius transformation is in the group G_m. Here is some of the relevant material.

Theorem 10. *A Möbius transformation* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is in* G_m *if and only if*

$$A/C = P_n/Q_n = (r_0\lambda, -1/r_1\lambda, \cdots, -1/r_n\lambda),$$

i.e., A/C *is a finite* λ*-fraction. [Ros54]*

This theorem has a corollary.

Corollary 11. *If* $A \in G_m$*, then* A *has one of two forms:*

$$A = \begin{pmatrix} a & b\lambda \\ c\lambda & d \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a\lambda & b \\ c & d\lambda \end{pmatrix},$$

where a, b, c, d *are polynomials in* λ^2 *with rational integral coefficients. [Ros54]*

He later proves the following.

Theorem 12. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, *and* $a, b, c, d \in \mathbb{Z}[\lambda]$*. Then* $A \in G_m$ *if and only if*

$$\frac{b}{d} = r_0\lambda + \cdots + \frac{\epsilon_n}{r_n\lambda} = \frac{P_n}{Q_n}, \frac{a}{c} = \frac{P_{n-1}}{Q_{n-1}} \text{ (the word ends in } T^{r_n}\text{), or}$$
$$\frac{a}{c} = r_0\lambda + \cdots + \frac{\epsilon_n}{r_n\lambda} + \frac{\epsilon_{n+1}}{r_{n+1}\lambda}, \frac{b}{d} = \frac{P_n}{Q_n}, \text{ (the word ends in } S.)$$

If the former, then A *has the form*

$$\begin{pmatrix} eP_{n-1} & P_n \\ eQ_{n-1} & Q_n \end{pmatrix}.$$

If the latter, then A *has the form*

$$\begin{pmatrix} eP_n & P_{n-1} \\ eQ_n & Q_{n-1} \end{pmatrix}.$$

$e = \pm 1$ *is chosen so that* $\det A = 1$*, and* $\epsilon_i = \pm 1$*. [Ros86]*

Rosen's work turned out to not be useful at all. His theorems, while they do provide a method for determining whether or not a given Möbius transformation is an element of G_5, do not index the elements of G_5 in a systematic manner that enables one to perform the double coset sum that is necessary for finding the Fourier coefficients we seek. Therefore, I made one last attempt, using a method derived from Leutbecher's work.

Leutbecher and the Height Function

It was known that $\mathbb{Z}[\lambda]$ is a Euclidean domain. Leutbecher showed the following ([Leu67],[Leu74]) :

Theorem 13. *Let $\alpha \in \mathbb{Q}(\lambda)$. Then there exists $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_5$ such that $\frac{b}{d} = \alpha$.*

John Leo ([Leo08]) shows that this implies that given an element $\alpha \in \mathbb{Q}[\lambda]$, there is an $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_5$ such that $\frac{d}{c} = \alpha$, and that if there are two such matrices, then they have the same bottom row (up to a factor of ±1.) He then goes on to apply the result to the double coset formula. With some work, he shows that we can write the Fourier coefficients of the Eisenstein series as

$$a_n = \left(\frac{2\pi}{\lambda i}\right)^k \frac{n^{k-1}}{(k-1)!} \left(1 + (-1)^n (2\lambda^2)^{-k} + \sum_{0<\alpha<\lambda/2} h(\alpha)^{-k} 2\cos(2\pi n\alpha/\lambda)\right)$$

where $h(\alpha) = c$ is the height function and d/c is the unique representative of α in $(-\lambda/2, \lambda/2]$. This means the problem of finding the Fourier coefficients has been reduced to figuring out how to find $h(\alpha)$ when given an α.

I had no idea how one would find the height function, so I just began by finding values for simple cases. I made several such calculations, but without ever finding a general method. By this time, the summer was almost over, so I had to begin writing this report.

Conclusion

While I failed to solve the problem I was assigned, I still feel my summer REU project was an excellent experience, and I got a great deal out of it.

I learned quite a bit of math on my own. More importantly, I gained skills and experience that will help me with further mathematical research. I learned the importance of researching a problem before trying to work on it. I learned that one does not always succeed in their research, and I also learned the effort and perseverance necessary to work on a problem for so long, without results. Also, because of this report, I was forced to learn how to write in Latex, something I had been putting off learning for quite a while. Surprisingly, I realized that it was much simpler than I had expected, and I learned the program very quickly.

Despite having learned these very worthwhile lessons, the chief benefit of the REU, in my opinion, is that I was exposed to a lot of math that I normally would not have been, being an undergraduate. I perhaps would not have even seen this material until I was a second year graduate student. Thus, I have a clearer understanding of what my mathematical interests are, and what areas of math I might one day like to work in. In addition, because my problem was related to John Leo's thesis, I was given a view into math outside of the classroom, and I now have a much better idea of what actual mathematical research is like; this is helpful, as an undergraduate student's mathematical experience is nearly all in the classroom.

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