Math 132
Homework 6 Comments

I graded 4 of the problems:
Section 5.3: 1e
Section 5.4: 1d
Section 5.6: 1
Section 5.7: 1b

Each problem is worth 3 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 3 indicates a correct or nearly correct solution. Otherwise the grade given is 1 or 2 depending upon how much work was put in and how close the solution is to being correct. How neat and clear your solution is also affects the grade I give.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and return the homework in the next section.

The following are comments and occasionally solutions for the graded problems.

General Comments
The maximum number of points was 12. The high score was 12, the median was 7 and the mean was 7.8. As this was a fairly short and easy assignment, and on top of that the answers to every problem were in the back of the book, I hoped everyone would get a 12 this time, but it was not to be. I was fairly strict in grading, and only two people got a 12, although there were several 11's (many people missed a point on 5.4 1d). Please make sure you understand any subtleties of problems you may have missed.

5.3 1. I graded part (e). This problem is easy if you use the ratio test. First let $w = z^2$ and $a_k = \frac{2^k}{k^2+k}$. Then we have a power series $\sum a_k w^k$. By the ratio test the radius of convergence is $\lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \left|\frac{2^k}{k(k+1)} \cdot \frac{(k+1)(k+2)}{2^{k+1}}\right| = \lim_{k\to\infty} \left|\frac{k+2}{2k}\right| = \frac{1}{2}$. Thus the sequence converges for $\left|z^2\right| < 1/2$ or $|z| < 1/\sqrt{2}$. Some people did a substitution $w = 2z^2$ and then seemed to think this was a geometric series that converges for $|w| < 1$ but this is not true–you still need to use the ratio test. Other people tried to use the root test but didn't justify the limits of the roots. I gave only one point for these kinds of attempts.

5.4 1. Problem 2 was the interesting one of this section but we went over it in discussion so I graded part (d) of problem 1. To get full credit, you had to be clear about where $\operatorname{Log} z$ fails to be analytic, which is $(-\infty, 0]$. Since 0 is the closest such point to $1 + 2i$, we have the radius of convergence $R = \sqrt{5}$. Those who said the singularity was just 0 got two points, as did those who said things like the function is not defined for $z < 0$, which makes no sense since you can't order complex numbers. Those who offered no explanation at all got at most one point.

5.6 1. There is a typo in the solution of this problem, which was a good opportunity to see if people were trying or not. If you did minimal work and just copied down the solution, I gave at most one point. There is a correction for the solution in the errata for Gamelin's book; a pointer to this errata can be found on my class web page.

Following the example in the book, we write

$$\begin{aligned}
\frac{1}{\cos z} &= \frac{1}{1 - (z^2/2!) + (z^4/4!) - (z^6/6!) + O(z^8)} \\
&= 1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + O(z^8)\right) + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + O(z^8)\right)^2 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + O(z^8)\right)^3 + O(z^8) \\
&= 1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!}\right) + \left(\frac{z^4}{2!2!} - 2\frac{z^6}{4!2!}\right) + \left(\frac{z^6}{2!2!2!}\right) + O(z^8) \\
&= 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \frac{61}{720}z^6 + O(z^8).
\end{aligned}$$

When you multiply out you only need to keep the terms of order seven or less, which of course end up being terms of order six or less since there are no odd terms in the power series expansion of $\cos z$. The others get absorbed into the $O(z^8)$.

5.7 1. I graded part (b); the other two parts we discussed in section. Write $\frac{1}{z} + \frac{1}{z^5}$ as $\frac{z^4+1}{z^5}$. Since 0 is not a root of the numerator, it follows there are four zeros of this function (by the fundamental theorem of algebra, recently proved!), namely the roots of $z^4 + 1$. There are several ways to find the roots of this polynomial. One is to note that they must satisfy $z^4 = -1$, and that $-1 = e^{\pi i + 2\pi k i}$, so that the general solution is $z = e^{\pi i/4 + k\pi i/2}$; then check that $e^{\pm \pi i/4}, e^{\pm 3\pi i/4}$ are the only four distinct solutions of this. Another approach is to first factor $z^4 + 1 = (z^2 + i)(z^2 - i)$ and find the two square roots of each of $i, -i$ by the same method.

Here's a geometric way to solve the problem that I like. Note that $(z^4 + 1)(z^4 - 1) = z^8 - 1$, the roots of which satisfy $z^8 = 1$; in other words the 8th roots of unity. These are simply $e^{k\pi i/4}$ for $0 \leq k < 8$; you can best visualize them as the vertices of an octagon inscribed in the unit circle. Now note that these are 8 distinct roots, and that four of them, namely $\pm 1, \pm i$, are the roots (fourth roots of unity) of $z^4 - 1$, so the other four must be the roots of $z^4 + 1$.

To prove they are simple roots, calculate the derivative of $\frac{z^4+1}{z^5}$ which is $\frac{z^5(4z^3) - 5z^4(z^4+1)}{z^{10}} = \frac{-z^4+1}{z^6}$, so the roots the numerator are the aforementioned roots of $z^4 - 1$ which are distinct from those of $z^4 + 1$. I was generous in grading this problem and generally gave full or nearly full credit if it looked like you tried.