Calculus Theory 1 Challenge Problems

Spring 2010

April 28, 2010

Instructions

When writing a solution, indicate both the chapter number (or "Misc" for miscellaneous problems) and the problem number (the problem number on this sheet, not the Spivak problem number). For multi-part problems, it is fine to solve just some of the parts. The number of points each problem or part is worth is given in square brackets; e.g. "[5]" means 5 points. A problem of the form "Spivak x" means it is from the text (4th edition), problem x in the appropriate chapter.

Points you earn over 100 points count as extra credit. For each 3 points over 100 points you get 1% toward your total semester quiz grade. When that reaches 100% you may apply extra credit to your total semester problem set grade at the same rate of 3 points per 1%. When both quiz and problem set grades reach 100% you may not earn any additional extra credit for challenge problems, although you are of course welcome to do more! All challenge problems must be turned in by May 20, 2010.

Miscellaneous Problems

1. Show that there is a 1-1 correspondence between fractional linear transformations $\frac{ax+b}{cx+d}$ and 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as follows. Let $f(x) = \frac{ax+b}{cx+d}$, $g(x) = \frac{Ax+B}{Cx+D}$, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. If $f(g(x)) = \frac{\alpha x+\beta}{\gamma x+\delta}$, then show that $MN = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. [15]

2. Let
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
Both of these matrices are in $SL_2(\mathbb{Z})$, which is the special linear group of 2×2 matrices with integer coefficients which have determinant 1. Prove that any matrix in $SL_2(\mathbb{Z})$ is generated by S and T (meaning it is a matrix product of a number of S, T, S^{-1} and T^{-1} matrices). Hint: Start with a matrix M in $SL_2(\mathbb{Z})$ and see if you can get the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ from M by multiplying it repeatedly (on the left or right as necessary) by S, S^{-1}, T and/or T^{-1}. [50]

3. Find two functions f and g that are not constant multiplies of each other that satisfy $f' = g$ and $g' = f$. [30]

4. Solve any of the problems from the handout "Complex Numbers, Polynomials, and Symmetry". ([1] per problem; max 30 points)

5. Find a function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(f(x)) = -x$ for all $x \in \mathbb{R}$. [50]

Chapter 1

1. Find a subfield of $\mathbb{R}$ that is not $\mathbb{R}$ or $\mathbb{Q}$ and prove it is a subfield (meaning that it contains 0 and 1 and is closed under addition, multiplication, and taking inverses). [20]

Chapter 2

1. Spivak 3(e). (Each part is worth [5].)
2. Spivak 27 and 28 (must be done together). [50]

Chapter 3

1. Spivak 11. ([5] each for parts (a)-(c); [10] each for (d) and (e))
2. Spivak 17. ([5] for each part)
3. Spivak 18. [20]
4. Spivak 19. ([10] per part)
5. Spivak 21. ([3] per part)
6. Spivak 22. ([5] per part)
7. Spivak 23. ([5] per part)
8. Spivak 24. ([10] per part)
9. Spivak 25. [15]
10. Spivak 26. [10]
11. Spivak 27. ([5] per part)
12. Spivak 28. ([5] per part except (c) which is worth [10])

Chapter 4

1. Spivak 7. ([5] per part)
2. Spivak 8. ([5] per part)
3. Spivak 17. ([3] per part)
4. Spivak 18. ([3] per part)

Chapter 4, Appendix 1 (Vectors)

1. Spivak 1. ([5] per part)
2. Spivak 2. ([5] per part)
3. Spivak 3. ([5] per part)
4. Spivak 4. [10]
5. Spivak 5. ([5] per part)
6. Spivak 6. [10]
7. Spivak 7. [10]

Chapter 4, Appendix 3 (Polar Coordinates)

1. Spivak 1. [10]
2. Spivak 2. [10]
3. Spivak 3. ([3] per part)
4. Spivak 4. ([3] per part)

Chapter 5

1. Spivak 3(ii–vi). ([10] per part)
2. Spivak 4. ([2] per part)
3. Spivak 7. [10]
4. Spivak 8. ([5] per part)
5. Spivak 9. [5]
6. Spivak 10. ([10] per part)
7. Spivak 15. ([5] per part)
8. Spivak 16. ([10] per part)
9. Spivak 17. ([5] per part)
10. Spivak 18. ([10] per part)
11. Spivak 19. [10]
12. Spivak 20. [15]
13. Spivak 21. ([5] per part)

14. Spivak 22. [10]
15. Spivak 25. ([3] per part)
16. Spivak 26. ([10] per part)
17. Spivak 27. ([3] per part)
18. Spivak 31. [15]
19. Spivak 34. [5]
20. Spivak 35. ([5] per part)
21. Spivak 36. ([5] per part, including the definition)
22. Spivak 38. ([5] per part)

Chapter 6

1. Spivak 6. ([5] per part)
2. Spivak 9. ([10] per part)

Chapter 7

1. Spivak 7. [5]
2. Spivak 12. ([5] per part)

Chapter 8

1. Spivak 3. ([10] for (a), [20] for (b))
2. Spivak 11. ([5] for each (a)–(c), [20] for (d))

Chapter 9

1. Spivak 13. [15]
2. Spivak 14. [10]
3. Spivak 15. ([10] per part)
4. Spivak 16. [10]
5. Spivak 19. ([10] per part)
6. Spivak 20. ([10] per part)

7. Spivak 21. ([3] per part)
8. Spivak 23. [5]
9. Spivak 24. [5]
10. Spivak 25. [5]
11. Spivak 26. ([2] per part)
12. Spivak 27. [15]
13. Spivak 28. ([10] per part)
14. Spivak 29. [15]
15. Spivak 30. ([2] per part)

Chapter 10

1. Spivak 2. ([3] for each part)
2. Spivak 11. [15]
3. Spivak 13. ([5] for (a), [10] for (b))
4. Spivak 17. [10]
5. Spivak 18. ([3] for (a) and (b), [5] for (c))
6. Spivak 19. ([10] per part)
7. Spivak 20. [20]
8. Spivak 23. ([5] per part)
9. Spivak 25. [5]
10. Spivak 26. ([10] per part)
11. Spivak 27. ([10] per part)
12. Spivak 28. [10]
13. Spivak 29. [15]
14. Spivak 30. [20]
15. Spivak 31. [10]
16. Spivak 32. ([10] per part)

Chapter 11

1. Spivak 4. ([3] for (a); [10] for (b) and (c))
2. Spivak 6. [10]
3. Spivak 8. ([10] per part)
4. Spivak 14. [15]
5. Spivak 15. [15]
6. Spivak 17. [10]
7. Spivak 18. [20]
8. Spivak 19. ([8] per part)
9. Spivak 20. [20]
10. Spivak 21. [2]
11. Spivak 22. [5]
12. Spivak 25. ([5] per part)
13. Spivak 26. [10]
14. Spivak 38. [10]
15. Spivak 39. [10]
16. Spivak 41. ([10] per part)
17. Spivak 57. [10]
18. Spivak 65. [10]
19. Spivak 66. ([5] per part)

Chapter 12

1. To be determined.